



## On strong cellularity type properties of Lindelöf groups

D. Buhagiar<sup>a,\*</sup>, B. Pasynkov<sup>b</sup>

<sup>a</sup> *Department of Mathematics, University of Malta, Msida MSD.06, Malta*

<sup>b</sup> *General Topology and Geometry, Mechanics and Mathematics Faculty, Moscow State University,  
119899 Moscow, Russia*

Received 6 September 2004; received in revised form 30 November 2004; accepted 2 December 2004

---

### Abstract

We prove several facts about cellularity and  $\kappa$ -cellularity of  $\lambda$ -Lindelöf groups generated by their  $\kappa$ -stable subspaces. For example, if a Lindelöf group  $G$  is generated by its  $\kappa$ -stable subspace then  $\kappa$ -cellularity (and hence cellularity) of  $G$  does not exceed  $\kappa$ . In particular,  $\omega_1$ -cellularity (and hence cellularity) of a Lindelöf group does not exceed  $\omega_1$  if this group is generated by its  $\omega_1$ -Lindelöf subspace which is a  $P$ -space. For any cardinal  $\mu$  with  $\omega < \mu \leq \mathfrak{c}$  a Lindelöf group  $G$  is constructed which is separable (and hence has countable cellularity) while  $\omega$ -cellularity of  $G$  is equal to  $\mu$ .

© 2004 Elsevier B.V. All rights reserved.

MSC: 54H11; 54A25; 54B35; 54D20

Keywords: Lindelöf group;  $(\tau-)$  cellularity;  $(\tau-)$  stability

---

We consider Tychonoff spaces (spaces, for short) and Hausdorff (and hence Tychonoff) topological groups only. The word “map” is used to denote a continuous mapping between spaces. Everywhere below  $\lambda$ ,  $\mu$  and  $\tau$  are infinite cardinal numbers.

---

\* Corresponding author.

E-mail addresses: [david.buhagiar@um.edu.mt](mailto:david.buhagiar@um.edu.mt) (D. Buhagiar), [pasynkov@mech.math.msu.su](mailto:pasynkov@mech.math.msu.su) (B. Pasynkov).

## 1. Introduction

It is well known that cellularity (i.e., the Souslin number)  $c(X)$  of compacta (i.e., compact Hausdorff spaces)  $X$  may be arbitrarily large but cellularity of any compact group is countable. Tkačenko [9] proved that cellularity of every  $\sigma$ -compact group is also countable. In particular, it is true for groups generated by their compact subspaces (for example, for the free topological groups of compacta). Since  $\sigma$ -compact groups are Lindelöf and Lindelöfness is the nearest property of compactness type to that of compactness, it was natural (after Tkačenko's result) to find out the behavior of cellularity in the class of Lindelöf groups.

The situation for Lindelöf groups is much more complicated than for compact ones. Generalizing Tkačenko's result on  $\sigma$ -compact groups, Uspenskii [12] showed that  $c(G) \leq \omega$  for any Lindelöf  $\Sigma$ -group  $G$  (a space  $X$  is called *Lindelöf  $\Sigma$*  if it is a continuous image of a perfect preimage of a separable metrizable space). Tkačenko [10] and Uspenskii [12] proved that  $c(G) \leq 2^\tau$  for any  $\tau$ -Lindelöf group  $G$  (a space  $X$  is called  *$\tau$ -Lindelöf* if every open cover of  $X$  has a subcover of cardinality  $\leq \tau$ ). Tkačenko [10] constructed a Lindelöf group  $G$  with  $c(G) = \omega_1$ . In [3], a Lindelöf group  $G$  is presented with  $c(G) = \mathfrak{c}$ . The following problem arose in connection with the results cited above.

**Problem 1.** When is

- (a)  $c(G) \leq \omega$ ;
- (b)  $c(G) \leq \omega_1$ ,

for a Lindelöf (and for an arbitrary) group  $G$ ?

A partial answer is contained in Uspenskii's paper [13] where the notion of  $\tau$ -stability of a space is used. Recall that a space  $X$  is called  *$\tau$ -stable* [1] if  $\text{nw}(Y) \leq \tau$  for any space  $Y$  which is a continuous image of  $X$  and has a *condensation* (i.e., a one-to-one map) onto a space of weight  $\leq \tau$ . It is completely natural to consider  $\tau$ -stability in the case of topological groups because any  $G_\tau$ -neighbourhood (i.e., a  $G_\tau$ -set) containing the identity element of an arbitrary topological group  $G$  contains a closed subgroup  $H$  such that  $G/H$  has a condensation onto a space with a  $\tau$ -locally finite base (which will have the weight  $\leq \tau$  if  $G$  is  $\tau$ -Lindelöf or  $c(G) \leq \tau$ ). In [13] (see Theorems 4, 5 and Corollary 1), Uspenskii gave an upper bound for the cellularity of a  $\lambda$ -Lindelöf group supposing that it is generated by a  $\mu$ -stable subspace. In particular, he stated that  $c(G) \leq \omega_1$  for any Lindelöf group  $G$  generated by its  $\omega$ -stable subspace. He also showed that the last inequality cannot be strengthened.

Sometimes it is more productive to consider more strong cardinal functions than cellularity in the theory of topological groups. Recall the corresponding definitions.

A family  $\eta$  of subsets of a space  $X$  is called *dense in a family  $\xi$*  of subsets of  $X$  if every element of  $\eta$  is contained in some element of  $\xi$  and  $\text{cl} \cup \eta = \text{cl} \cup \xi$ . Evidently, the inequality  $c(X) \leq \tau$  is equivalent to the property: any family  $\xi$  of open sets in  $X$  has a dense in  $\xi$  subfamily of cardinality  $\leq \tau$ . This observation explains the following definition (and the term “ $\tau$ -cellularity”).

Recall that:

- a  $G_\tau$ -set (= a set of type  $G_\tau$ ) in a space  $X$  is the intersection of at most  $\tau$ -many open sets in  $X$ ;
- a  $G_\tau$ -family in a space  $X$  is a family of  $G_\tau$ -sets in  $X$ ;
- the  $\tau$ -cellularity  $\text{cel}_\tau(X)$  of a space  $X$  is  $\max(\omega, \min\{\lambda: \text{any } G_\tau\text{-family } \xi \text{ in } X \text{ has a dense in } \xi \text{ subfamily of cardinality } \leq \lambda\})$ .

It seems that the definition of  $\text{cel}_\tau X$  was given by M.G. Tkačenko. The case of  $\text{cel}_\omega X$  was considered, for example, in [13].

Evidently,

$$c(X) \leq \text{cel}_\lambda(X) \leq \text{cel}_\mu(X) \quad \text{for } \lambda \leq \mu.$$

Note that Uspenskii [13] proved the inequality  $\text{cel}_\omega(G) \leq \omega$  for Lindelöf  $\Sigma$ - (in particular, for  $\sigma$ -compact) groups and Pasynkov [6] stated the inequality  $\text{cel}_\tau(G) \leq 2^\tau$  for  $\tau$ -Lindelöf groups  $G$ . These results strengthen Tkačenko's and Uspenskii's ones on  $c(G)$  cited above. In this paper, a strengthening of Uspenskii's results from [13] will be obtained and some additional information will be given even for  $c(G)$  of some Lindelöf groups  $G$ .

The following problems may be formulated in connection with the cited above results and Problem 1.

**Problem 2.** When is

- (c)  $\text{cel}_\omega(G) \leq \omega$ ;
- (d)  $\text{cel}_\omega(G) \leq \omega_1$ ,

for a Lindelöf (and for any) group  $G$ ?

**Problem 3.** Is it true that  $c(G) = \text{cel}_\omega(G)$  for all Lindelöf groups  $G$ ?

A complete answer to Problem 3 will be obtained in this paper. It will be shown that, for Lindelöf groups, the difference between  $c(G)$  and  $\text{cel}_\omega(G)$  may be arbitrary (in possible limits).

Often the cardinal function  $\tau - \text{cl}(X) = \min\{\mu: \text{for any } G_\tau\text{-family } \xi \text{ in } X, \text{ the closure } \text{cl}(\bigcup \xi) \text{ is a } G_\mu\text{-set in } X\}$  is considered together with  $\text{cel}_\tau X$ .

Evidently,

$$\lambda - \text{cl}(X) \leq \mu - \text{cl}(X) \quad \text{for } \lambda \leq \mu.$$

Note that the inequality  $\omega - \text{cl}(X) \leq \omega$  which implies, for example, the almost perfect normality of  $X$  [2], is useful in examinations of inductive dimensions [6] and is often proved together with the inequality  $\text{cel}_\omega(X) \leq \omega$ . The last inequality is stated by Uspenskii in [12] for Lindelöf  $\Sigma$ -groups. Some upper bounds for  $\omega - \text{cl}(G)$  for special Lindelöf groups will be given in this paper.

The proof of the main result of Section 2 (Theorem 1) is based on considering the inverse spectrum (= inverse system)  $S_G$  defined by a topological group  $G$ , which consists

of all quotient-spaces  $G/H_\alpha$ , where  $H_\alpha$  is a closed subgroup of  $G$ . Generally speaking, this spectrum is not continuous (i.e., for some well-ordered by inclusion family of closed subgroups  $H_\alpha, \alpha \in A$ , of  $G$  and for  $H = \bigcap \{H_\alpha: \alpha \in A\}$  the space  $G/H$  may not be the limit of the inverse spectrum of the spaces  $G/H_\alpha$ ), but sometimes this problem may be taken care of (see, for example, [6,7]). In this paper, the continuity of  $S_G$  is stated in some places.

## 2. Cellularity and $\tau$ -cellularity of $\lambda$ -Lindelöf groups for $\lambda < \tau$

Recall that  $\text{iw}(X) \leq \tau$  for a space  $X$  if  $X$  has a condensation onto a space of weight  $\leq \tau$ .

**Definition 1.** For cardinal numbers  $\lambda$  and  $\mu$ ,  $\lambda \leq \mu$ , a space  $X$  will be called  $(\lambda, \mu)$ -stable if  $\text{nw}(Y) \leq \mu$  for any continuous image  $Y$  of  $X$  with  $\text{iw}(Y) \leq \lambda$ .

Note that the  $(\lambda, \lambda)$ -stability coincides with  $\lambda$ -stability.

**Definition 2.** A topological group  $G$  will be called (strongly) algebraically  $(\lambda, \mu)$ -stable,  $\lambda \leq \mu$ , if for any closed normal subgroup (for any closed subgroup)  $N$ ,  $\text{iw}(G/N) \leq \lambda$  implies  $\text{nw}(G/N) \leq \mu$ . A group will be called (strongly) algebraically  $\lambda$ -stable if it is (strongly) algebraically  $(\lambda, \lambda)$ -stable.

**Lemma 1.** If a topological group is generated by a  $(\lambda, \mu)$ -stable subspace then it is algebraically  $(\lambda, \mu)$ -stable.

**Proof.** Assume that a group  $G$  is generated by its  $(\lambda, \mu)$ -stable subspace  $A$  and  $H$  is a closed normal subgroup of  $G$  such that  $\text{iw}(G/H) \leq \lambda$ ; let  $p: G \rightarrow X = G/H$  be the canonical map. Then, for  $B = pA$  we have  $\text{iw}(B) \leq \lambda$  and so  $\text{nw}(B) \leq \mu$ . Since  $\text{nw}(B^n) \leq \mu$ ,  $n = 2, 3, \dots$ , and  $G/H$  is a continuous image of a countable discrete union of powers  $B^n$ , we have  $\text{nw}(G/H) \leq \mu$ .  $\square$

Recall that an inverse spectrum  $S = \{X_\alpha, p_{\beta\alpha}; A\}$  is called  $\lambda$ -continuous for an infinite cardinal number  $\lambda$  (see, for example, [6]) if, for any monotone mapping  $j: \lambda \rightarrow A$ , there exists  $\gamma = \sup j\lambda$  in  $A$  and  $\Delta\{p_{\gamma j(\theta)}: \theta \in \lambda\}$  is a homeomorphism onto the limit of the inverse spectrum  $\{X_{j(\theta)}, p_{j(\kappa)j(\theta)}; \theta \in \lambda\}$ .

The following proposition is a generalization of Theorem 4 from [13] (for a slightly stronger notion of  $\lambda$ -continuity than in [13]).

**Proposition 1.** Assume that  $\lambda < \tau$  and a  $\lambda$ -Lindelöf group  $G$  is generated by a subspace  $X$  which is  $(\mu, \tau)$ -stable for all  $\mu$  with  $\lambda \leq \mu < \tau$ . Then  $G$  is the limit of a  $\lambda^+$ -continuous  $\tau$ -directed inverse spectrum  $S = \{G_\alpha, p_{\beta\alpha}; A\}$  of topological groups such that:

- (1)  $\text{nw}(G_\alpha) \leq \tau$  for all  $\alpha \in A$ ;
- (2) the limit homomorphisms  $p_\alpha: G \rightarrow G_\alpha$  are open.

We need the following lemma to prove Proposition 1.

**Lemma 2.** *Suppose that  $G$  is a  $\lambda$ -Lindelöf group and a family  $\eta = \{N_\alpha: \alpha \in A\}$  of closed normal subgroups of  $G$  is  $\lambda$ -directed (assuming also that  $\alpha < \beta$  if  $N_\beta \subset N_\alpha$ ); let  $N = \bigcap \eta$ . Then*

- (1) *for any neighbourhood  $O$  of the identity  $e$  of  $G$ , there exists a neighbourhood  $U$  of  $e$  and  $\alpha \in A$  such that  $U \cdot N_\alpha \subset O \cdot N$ ;*
- (2) *the topological group  $X = G/N$  is the limit of the inverse spectrum of topological groups  $S = \{X_\alpha = G/N_\alpha, p_{\beta\alpha}; A\}$ , where  $p_{\beta\alpha}$  is the canonical map of  $X_\beta$  onto  $X_\alpha$  for  $\beta > \alpha$ .*

**Proof.** Let  $p: G \rightarrow X$  and  $f_\alpha: X \rightarrow X_\alpha$  be the canonical maps. Evidently,  $X$  is  $\lambda$ -Lindelöf.

Take a neighbourhood  $O$  of  $e$ . There exists a neighbourhood  $U$  of  $e$  such that  $U^2 \subset O$ . The family  $\xi = \{G \setminus N_\alpha: \alpha \in A\} \cup \{U \cdot N\}$  is an open cover of  $G$ . Since  $G$  is  $\lambda$ -Lindelöf, there exists a subcover  $\{N_\alpha: \alpha \in B\} \cup \{U \cdot N\}$  of  $\xi$  with  $B \subset A$ ,  $|B| \leq \lambda$ . Since  $\eta$  is  $\lambda$ -centered, there exists  $\beta \in A$  such that  $\beta > \alpha$  for any  $\alpha \in B$ . Then  $N_\beta \subset \bigcap \{N_\alpha: \alpha \in B\} = G \setminus \bigcup \{(G \setminus N_\alpha): \alpha \in B\} \subset U \cdot N$ . Hence  $U \cdot N_\beta = U^2 \cdot N \subset O \cdot N$ , so (1) is proved.

Let  $Y$  be the limit of  $S$  and  $p_\alpha: Y \rightarrow X_\alpha$ ,  $\alpha \in A$ , be its projections. Since  $f_\alpha = p_{\beta\alpha} \circ f_\beta$  for  $\beta > \alpha$ , a continuous homomorphism  $f: X \rightarrow Y$  is defined so that  $f_\alpha = p_\alpha \circ f$ ,  $\alpha \in A$ . Since  $N = \bigcap \eta$ , the map  $f$  is a monomorphism. If  $y = \{y_\alpha\}_{\alpha \in A} \in Y$  then the sets  $f^{-1}y_\alpha$  are closed in  $X$  and their family is  $\lambda$ -centered (because  $A$  is  $\lambda$ -directed). It follows from the  $\lambda$ -Lindelöfness of  $X$  that  $f^{-1}y = \bigcap \{f_\alpha^{-1}y_\alpha: \alpha \in A\} \neq \emptyset$ . Hence  $f$  is an epimorphism. Finally, (1) implies that  $f$  is open and so we may identify  $X$  and  $Y$  by means of  $f$ .  $\square$

**Proof of Proposition 1.** We may suppose (see [4]) that  $G$  is a subgroup of the product  $\Pi$  of topological groups  $G_i$  of weight  $\leq \lambda$ ,  $i \in I$ . Let  $A = \{\alpha \subset I: |\alpha| \leq \tau\}$ ;  $pr_\alpha$  be the restriction to  $G$  of the projection of the product  $\Pi$  onto the subproduct  $\Pi_\alpha = \prod \{G_i: i \in \alpha\}$ ,  $N_\alpha = pr_\alpha^{-1}e_\alpha$  (where  $e_\alpha$  is the identity of the group  $\Pi_\alpha$ ),  $G_\alpha = G/N_\alpha$ ,  $\pi_\alpha$  be the canonical map of  $G$  onto  $G_\alpha$ ,  $\alpha \in A$ ;  $p_{\beta\alpha}$  be the canonical map of  $G_\beta$  onto  $G_\alpha$  for  $\beta > \alpha$  (i.e., for  $\beta \supset \alpha$ ),  $\alpha, \beta \in A$ . Evidently, we have a  $\tau$ -directed inverse spectrum of topological groups  $S = \{G_\alpha, p_{\beta\alpha}; A\}$ . Let  $H$  be the limit of  $S$  and  $p_\alpha: H \rightarrow G_\alpha$  be the limit projections,  $\alpha \in A$ . Evidently, for every  $\alpha$ , the group  $G_\alpha$  has a continuous monomorphism to the group  $\Pi_\alpha$ .

Let us prove (1).

Let  $|\alpha| = \nu$ . Take the case when  $\lambda \leq \nu < \tau$ . Then  $w(\Pi_\alpha) \leq \nu < \tau$  and, by Lemma 1,  $\text{nw}(G_\alpha) \leq \tau$ . Now let  $\nu = \tau$ . Take an injective and (strongly) monotone mapping  $j: \tau \rightarrow A$  such that  $j(\theta) < \alpha$  for all  $\theta \in \tau$  and  $\alpha = \bigcup \{j(\theta): \theta \in \tau\} = \sup \{j(\theta): \theta \in \tau\}$ . Then the family of all  $N_{j(\theta)}$ ,  $\theta \in \tau$ , is  $\lambda$ -directed and  $N_\alpha = \bigcap \{N_{j(\theta)}: \theta \in \tau\}$ . Since  $w(\Pi_{j(\theta)}) < \tau$ , by Lemma 1,  $\text{nw}(G_{j(\theta)}) \leq \tau$ ,  $\theta \in \tau$ . Hence, by Lemma 2,  $\text{nw}(G_\alpha) \leq \tau$ . We have proved (1).

Let us prove that  $S$  is  $\lambda^+$ -continuous.

Take a monotone mapping  $j: \lambda^+ \rightarrow A$ . Since  $\lambda^+ \leq \tau$ , we have  $\delta = \bigcup j(\lambda^+) \in A$  and  $\delta = \sup j(\lambda^+)$ . Let  $S_\delta = \{G_{j(\theta)}, p_{j(\kappa)j(\theta)}; \theta \in \lambda^+\}$ . If  $\delta \in j(\lambda^+)$  then, evidently, the group  $G_\delta$  is the limit of  $S_\delta$ . If  $\delta \notin j(\lambda^+)$  then  $|\delta| > \lambda$ . Hence the set  $B = \{\beta \subset \delta: |\beta| \leq \lambda\}$  is

$\lambda$ -directed and  $N_\delta = \bigcap \{N_\beta : \beta \in B\}$ . By Lemma 2,  $G_\delta$  is the limit of the inverse spectrum  $\Sigma_\delta = \{G_\beta, p_{\gamma\beta}; B\}$  and its limit projections coincide with the homomorphisms  $p_{\delta\beta}$ ,  $\beta \in B$ . The homomorphisms  $p_{\delta j(\theta)}$  define a continuous homomorphism  $i_\delta$  of  $G_\delta$  to the limit  $H_\delta$  of  $S_\delta$  such that  $p_{\delta j(\theta)} = p_{j(\theta)} \circ i_\delta$ , where  $p_{j(\theta)}$  is the limit projection of the spectrum  $S_\delta$ ,  $\theta \in \lambda^+$ . The surjectivity of all  $p_{\delta j(\theta)}$  implies the density of  $i_\delta G_\delta$  in  $H_\delta$ . Since, for any  $\beta \in B$ , there exists the minimal number  $\theta(\beta) \in \lambda^+$  such that  $\beta \subset j(\theta(\beta))$ , the continuous homomorphism  $k_\beta = p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\beta))} : H_\delta \rightarrow G_\beta$  is defined. Since the inequality  $\gamma > \beta$ , for  $\gamma, \beta \in B$ , implies the inequality  $\theta(\gamma) \geq \theta(\beta)$ , we convince ourselves that

$$\begin{aligned} k_\beta &= p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\beta))} = p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\gamma))j(\theta(\beta))} \circ p_{j(\theta(\gamma))} \\ &= p_{\gamma\beta} \circ p_{j(\theta(\gamma))\gamma} \circ p_{j(\theta(\gamma))} = p_{\gamma\beta} \circ k_\gamma. \end{aligned}$$

Hence a continuous homomorphism  $k : H_\delta \rightarrow G_\delta$  is defined such that  $p_{\delta\beta} \circ k = k_\beta$  for any  $\beta \in B$ . But since

$$p_{\delta\beta} \circ k \circ i_\delta = k_\beta \circ i_\delta = p_{j(\theta(\beta))\beta} \circ p_{j(\theta(\beta))} \circ i_\delta = p_{j(\theta(\beta))\beta} \circ p_{\delta j(\theta(\beta))} = p_{\delta\beta}$$

for any  $\beta \in B$ , we conclude that  $k \circ i_\delta$  is the identity map of  $G_\delta$ . The Hausdorffness of  $H_\delta$  and the density of  $i_\delta G_\delta$  in  $H_\delta$  imply that  $i_\delta$  and  $k$  are mutually inverse homeomorphisms. We have proved that  $S$  is  $\lambda^+$ -continuous.

Since  $\pi_\alpha = p_{\beta\alpha} \circ \pi_\beta$  for any  $\alpha, \beta \in A$ ,  $\alpha < \beta$ , a continuous homomorphism  $\pi : G \rightarrow H$  is defined such that  $\pi_\alpha = p_\alpha \circ \pi$ ,  $\alpha \in A$ . The epimorphness of all  $\pi_\alpha$  implies that  $\pi G$  is dense in  $H$ . Since  $G$  is a subspace of  $\Pi$ , we conclude that  $\pi$  is a topological embedding. Finally, for any point  $h \in H$ , the family  $h^* = \{\pi_\alpha^{-1} p_\alpha h : \alpha \in A\}$  is closed in  $G$  and is  $\tau$ -(and so  $\lambda$ -)centered. It follows from the  $\lambda$ -Lindelöfness of  $G$  that  $\bigcap h^* \neq \emptyset$ . Since  $\bigcap \{p_\alpha^{-1} p_\alpha h : \alpha \in A\} = \{h\}$ , we have  $h \in \pi G$ . Hence  $\pi$  is an isomorphism. The openness of  $\pi_\alpha$ , the surjectivity of  $\pi$  and the relation  $\pi_\alpha = p_\alpha \circ \pi$  give us the openness of  $p_\alpha$ .  $\square$

**Theorem 1.** Suppose that  $\lambda < \tau$  and a  $\lambda$ -Lindelöf group  $G$  is generated by a subspace  $X$  which is  $(\mu, \tau)$ -stable for all  $\mu$  with  $\lambda \leq \mu < \tau$ . Then

$$c(X) \leq \text{cel}_\omega(G) \leq \text{cel}_\tau(G) \leq \tau \quad \text{and} \quad \omega - \text{cl}(G) \leq \tau - \text{cl}(G) \leq \tau.$$

Moreover, for any  $G_\tau$ -family  $\xi$  in  $G$ , there exist a normal closed subgroup  $N$  of  $G$  and a closed (and  $G_\tau$ -) family  $\eta$  in  $H = G/N$  such that  $\text{nw}(H) \leq \tau$ ,  $|\eta| \leq \tau$  and the family  $p^{-1}\eta$  (where  $p$  is the canonical map of  $G$  onto  $H$ ) is dense in  $\xi$ .

**Proof.** Let  $\xi$  be a  $G_\tau$ -family in  $G$ .

Take an inverse spectrum  $S = \{G_\alpha, p_{\beta\alpha}; A\}$  with the properties such as in Proposition 1.

Since  $S$  is  $\tau$ -directed, we may find a family  $\zeta$  in  $G$  such that its elements are contained in elements of  $\xi$ ; for every  $F \in \zeta$  there exist  $\alpha = \alpha(F) \in A$  and a closed set  $\Phi = \Phi(F)$  in  $G_\alpha$  such that  $F = p_\alpha^{-1}\Phi$ ;  $\bigcup \zeta = \bigcup \xi$ . Suppose that there is no dense in  $\zeta$  subsystem of cardinality  $\leq \tau$ . Then, for any ordinal number  $\theta < \tau^+$ , there exist  $\alpha(\theta) \in A$ , a closed (and  $G_\tau$ -) set  $\Phi_\theta$  in  $G_{\alpha(\theta)}$  and a point  $x_\theta \in F_\theta = p_{\alpha(\theta)}^{-1}\Phi_\theta \in \zeta$  such that

$$x_\theta \in F_\theta \setminus (\text{cl} \cup \{F_\kappa : \kappa < \theta\}), \quad \theta < \tau^+. \quad (1)$$

Let  $C_\theta = \{x_\kappa : \kappa < \theta\}$ ,  $2 \leq \theta < \tau^+$ ;  $C = \{x_\theta : \theta < \tau^+\}$ .

The  $\tau$ -directedness of  $S$  allows us to suppose that

$$\alpha(\kappa) \leq \alpha(\theta) \quad \text{for } \kappa < \theta < \tau^+. \quad (2)$$

Since  $\text{nw}(G_\alpha) \leq \tau$  for all  $\alpha \in A$ , there exists  $\theta(1) < \tau^+$  such that  $p_{\alpha(1)}C_{\theta(1)}$  is dense in  $p_{\alpha(1)}C$ . Let  $\delta(1) = \alpha(\theta(1))$ . It is possible, by means of transfinite induction and using the  $\tau$ -directedness of  $S$  and  $\tau^+$ , to choose  $\theta(k) < \tau^+$  and  $\delta(k) \in A$ ,  $k < \lambda^+$ , so that

$$\begin{aligned} \theta(k) < \theta(l) \quad \text{if } k < l < \lambda^+, \quad \delta(k) = \alpha(\theta(k)) \quad \text{and} \\ \text{cl } p_{\delta(k)}C_{\theta(k+1)} = \text{cl } p_{\delta(k)}C, \quad k < \lambda^+. \end{aligned} \quad (3)$$

Since  $\lambda^+ \leq \tau$ , we have  $\theta(\infty) = \sup\{\theta(k) : k < \lambda^+\} < \tau^+$ . It follows from (3) and (2) that the mapping  $\delta : \lambda^+ \rightarrow A$  is monotone. Hence, there exists  $\delta(\infty) = \sup\{\delta(k) : k < \lambda^+\}$  in  $A$ . The set  $p_{\delta(\infty)}C_{\theta(\infty)}$  is dense in  $p_{\delta(\infty)}C$  because  $S$  is  $\lambda^+$ -continuous. Hence  $p_{\delta(\infty)}x_{\theta(\infty)} \in \text{cl } p_{\delta(\infty)}C_{\theta(\infty)}$ . It follows from the openness of  $p_{\delta(\infty)}$  that  $x_{\theta(\infty)} \in \text{cl } p_{\delta(\infty)}^{-1}p_{\delta(\infty)}C_{\theta(\infty)}$ . But (see (2))  $\delta(\infty) \geq \alpha(\kappa)$  for all  $\kappa < \theta(\infty)$ . Hence

$$\begin{aligned} x_{\theta(\infty)} \in \text{cl } p_{\delta(\infty)}^{-1}p_{\delta(\infty)}C_{\theta(\infty)} \subset \text{cl} \cup \{p_{\alpha(\kappa)}^{-1}p_{\alpha(\kappa)}x_\kappa : \kappa < \theta(\infty)\} \\ \subset \text{cl} \cup \{F_\kappa : \kappa < \theta(\infty)\}. \end{aligned}$$

But, by (1), this is impossible. Thus there exists a dense subfamily  $\zeta'$  of  $\zeta$  of cardinality  $\leq \tau$ .

Since  $S$  is  $\tau$ -directed, there exist an index  $\alpha \in A$  and a closed family  $\eta$  in  $H = G_\alpha$  such that  $\zeta' = p_\alpha^{-1}\eta$ . It follows from the openness of  $p_\alpha$  that  $\text{cl} \cup \xi = \text{cl} \cup \zeta = \text{cl} \cup \zeta' = p_\alpha^{-1} \text{cl} \cup \eta$ . Since  $\text{nw}(H) \leq \tau$ , we conclude that  $\text{cl} \cup \eta$  and  $\text{cl} \cup \xi$  are  $G_\tau$ -sets in  $H$  and in  $G$  respectively. Evidently, the required normal subgroup  $N$  of  $G$  is  $p_\alpha^{-1}e_\alpha$ , where  $e_\alpha$  is the identity of the group  $H$ .  $\square$

**Corollary 1.** *If a Lindelöf group  $G$  is generated by its  $(\omega, \omega_1)$ -stable (in particular,  $\omega$ - or  $\omega_1$ -stable) subspace then*

$$c(G) \leq \text{cel}_\omega(G) \leq \text{cel}_{\omega_1}(G) \leq \omega_1 \quad \text{and} \quad \omega - \text{cl}(G) \leq \omega_1 - \text{cl}(G) \leq \omega_1.$$

This corollary is a generalization and a strengthening of Corollary 2 in [13].

**Corollary 2.** *If a Lindelöf group  $G$  is generated by a  $\mu$ -stable subspace,  $\mu > \omega$ , then*

$$c(G) \leq \text{cel}_\omega(G) \leq \text{cel}_\mu(G) \leq \mu \quad \text{and} \quad \omega - \text{cl}(G) \leq \mu - \text{cl}(G) \leq \mu.$$

Recall that a space  $X$  is a  $P$ -space if every  $G_\delta$ -subset of  $X$  is open in  $X$ . Also recall that, for a discrete space  $X$  and an infinite cardinal number  $\tau$ , the one-point  $\tau$ -Lindelöfication  $L_\tau X$  is the disjoint union of  $X$  and a point  $l$  with the topology consisting of all subsets of  $X$  and all sets of type  $\{l\} \cup (X \setminus L)$ , where  $|L| \leq \tau$ . Evidently,  $L_\tau X$  is a  $\tau$ -Lindelöf  $P$ -space. Since every first countable continuous image of any  $\omega_1$ -Lindelöf  $P$ -space consists of not greater than  $\omega_1$  points, any  $\omega_1$ -Lindelöf  $P$ -space is  $(\omega, \omega_1)$ -stable. Thus we have the following.

**Corollary 3.** *If a Lindelöf group  $G$  is generated by a  $\omega_1$ -Lindelöf (in particular, Lindelöf) subspace which is a  $P$ -space then*

$$c(G) \leq \text{cel}_\omega(G) \leq \text{cel}_{\omega_1}(G) \leq \omega_1 \quad \text{and} \quad \omega - \text{cl}(G) \leq \omega_1 - \text{cl}(G) \leq \omega_1.$$

This corollary is a generalization and a strengthening of Corollary 3 in [13], Theorem 3.8 in [5] and Corollary 4.14 in [11].

### 3. Lindelöf groups with countable cellularity and arbitrary (possible) $\omega$ -cellularity

**Theorem 2.** *For any  $\mu$ ,  $\omega < \mu \leq \mathfrak{c}$ , there exists a Lindelöf separable group  $F_\mu$  with  $\text{nw}(F_\mu) = \mu$ ,  $\text{iw}(F_\mu) = \omega$  and  $(c(F_\mu) = \omega <) \text{cel}_\omega(F_\mu) = \mu$  which is  $(\omega, \mu)$ -stable but is not  $(\omega, \lambda)$ -stable for  $\lambda < \mu$ .*

**Proof.** Let  $R^2$  be the set of all points of the space  $\mathbb{R}^2$  and  $\mathcal{T}_e$  be the topology of  $\mathbb{R}^2$ .

Following Przymusiński [8, Corollary 4], represent  $\mathbb{R}^2$  as the union of two disjoint sets  $T_1$  and  $T_2$  so that, for any  $n \in \omega$  and any closed in  $(\mathbb{R}^2)^n$  set  $F$ , the relation  $F \cap (T_i)^n = \emptyset$  for  $i = 1$  or  $i = 2$  implies the relation  $|F|_n \leq \omega$  (i.e., there exists a countable set  $A \subset \mathbb{R}^2$  such that  $F \subset \bigcup \{(\mathbb{R}^2)_1 \times \cdots \times (\mathbb{R}^2)_{i-1} \times A \times (\mathbb{R}^2)_{i+1} \times \cdots \times (\mathbb{R}^2)_n : i = 1, \dots, n\}$ ).

Below  $\mathcal{T}_e^n$  denotes the topology of  $(\mathbb{R}^2)^n$ . Put  $R_i = \{(x, y) \in T_i : y = 0\}$ ,  $i = 1, 2$ . We may suppose that  $|R_2| = \mathfrak{c}$ . Let  $\mathcal{T}$  be a new topology on  $R^2$  a base of which is  $\mathcal{T}_e \cup (\{(a, 0)\} \cup \{(x, y) \in R^2 : (x - a)^2 + (y - \varepsilon)^2 < \varepsilon^2 : (a, 0) \in R_2, \varepsilon > 0\})$ . Put  $P = (R^2, \mathcal{T})$ .

For any  $\mu$ ,  $\omega < \mu \leq \mathfrak{c}$ , fix a subset  $A_\mu$  of  $R_2$  of cardinality  $\mu$  and let  $P_\mu$  be the set  $R^2 \setminus (R_2 \setminus A_\mu)$  with the topology induced by  $\mathcal{T}$  on it. Evidently, the discrete topology is induced on  $A_\mu$  in  $P_\mu$ . Hence  $\text{nw}(P_\mu) \geq \mu$ . Since the space  $P_\mu \setminus A_\mu$  has a countable base and  $|A_\mu| = \mu$ , we have  $\text{nw}(P_\mu) \leq \mu$ . Thus  $\text{nw}(P_\mu) = \mu$ . It follows from this that  $\text{nw}((P_\mu)^n) = \mu$  for any  $n \in \omega$ . Since the free topological group  $F_\mu = F(P_\mu)$  of the space  $P_\mu$  is a continuous image of a countable discrete union of all finite powers of the space  $P_\mu$  and  $F_\mu$  contains  $P_\mu$  as a subspace, we have  $\text{nw}(F_\mu) = \mu$ .

Evidently, the space  $P_\mu$  is separable. Hence all of its finite powers and the group  $F_\mu$  are also separable and so  $c(F_\mu) = \omega$ .

We now show that the space  $F_\mu$  is  $(\omega, \mu)$ -stable but not  $(\omega, \lambda)$ -stable for  $\lambda < \mu$ . The first assertion follows from the fact that the network weight of all continuous images of  $F_\mu$  is not greater than  $\mu$ . The second one follows from the inequality  $\text{nw}(F_\mu) \geq \mu$  and from the fact that  $F_\mu$  has a condensation onto a separable metrizable space. Indeed,  $P_\mu$  has a condensation onto a subset  $Q_\mu$  of  $\mathbb{R}^2$ . Hence  $F_\mu$  has a continuous isomorphism onto the free topological group  $G_\mu$  of  $Q_\mu$ . The existence of a countable base in  $Q_\mu$  implies the existence of a countable network in  $G_\mu$ . This allows to condense  $G_\mu$  onto a separable metrizable space. Hence  $\text{iw}(F_\mu) = \omega$ .

We next prove that  $F_\mu$  is Lindelöf. It is sufficient for this to prove that all finite powers of  $P_\mu$  are Lindelöf (the method of the proof is similar to one in [8]).

First, we shall prove that  $P_\mu$  itself is Lindelöf. Let  $\eta$  be its open cover. Topologies  $\mathcal{T}_e$  and  $\mathcal{T}$  coincide on  $T_1$ . Hence there exists a countable subfamily  $\zeta$  of  $\eta$  covering  $T_1$ . The



set  $\bigcup \zeta$  contains a neighbourhood  $O$  of  $T_1$  in the subspace  $R^2 \setminus (R_2 \setminus A_\mu)$  of the space  $\mathbb{R}^2$ . It follows from this (by the choice of the sets  $T_i$ ) that  $P_\mu \setminus O$  is countable. This allows to choose a countable subcover of  $\eta$ .

Suppose that all powers  $(P_\mu)^m$ ,  $1 \leq m < n$ , are Lindelöf. We show that  $X = (P_\mu)^n = \prod \{(P_\mu)_k : k = 1, \dots, n\}$  is also Lindelöf. Let  $\eta$  be an open cover of  $X$ . As above, there exists a countable subfamily of  $\eta$  covering some neighbourhood  $O$  of  $(T_1)^n$  in the subspace  $(R^2 \setminus (R_2 \setminus A_\mu))^n$  of the space  $(\mathbb{R}^2)^n$ . By the choice of the sets  $T_i$ , there exists a countable set  $C$  in  $P_\mu$  such that  $(P_\mu)^n \setminus O$  is contained in the union of subproducts  $\Pi_k$ ,  $k = 1, \dots, n$ , of the product  $X$  such that the factor of  $\Pi_k$  is contained in  $(P_\mu)_k$  is  $C$  and other factors coincide with the corresponding factors of  $X$ . Thus  $X \setminus O$  is contained in the countable union of subsets homeomorphic to  $(P_\mu)^{n-1}$  which are Lindelöf, by the inductive assumption. It follows from this that  $\eta$  has a countable subcover. We have proved that all finite powers of  $P_\mu$  and (so)  $F_\mu$  are Lindelöf.

Finally we prove the equality  $\text{cel}_\omega(F_\mu) = \mu$ . The inequality  $\text{cel}_\omega(F_\mu) \leq \mu$  follows from the relation  $\text{nw}(F_\mu) \leq \mu$ . All one-point sets in  $F_\mu$  are of type  $G_\delta$  because  $F_\mu$  may be condensed onto a (separable) metrizable space. Since  $P_\mu$  is a subspace of  $F_\mu$  and contains a discrete (in itself) subspace  $A_\mu$  of cardinality  $\mu$ , the  $G_\delta$ -family of all one-point subsets of  $A_\mu$  has no dense subfamily of cardinality  $< \mu$ .  $\square$

## References

- [1] A.V. Arhangel'skii, Factorization theorems and spaces of functions: Stability and monolithicity, Dokl. Akad. Nauk SSSR 265 (5) (1982) 1039–1043 (in Russian), also: Soviet Math. Dokl. 26 (1982) 177–181 (in English).
- [2] A.Ch. Chigogidze, On a generalization of perfectly normal spaces, Topology Appl. 13 (1982) 15–20.
- [3] P.M. Gartside, E.A. Reznichenko, O.V. Sipacheva, Mal'tsev and retral spaces, Topology Appl. 80 (1997) 115–129.
- [4] I.I. Guran, On topological groups close to finally compact groups, Dokl. Akad. Nauk SSSR 256 (6) (1981) 1305–1307 (in Russian), also: Soviet Math. Dokl. 23 (1981) 173–175 (in English).
- [5] G.L. Itzkowitz, V.V. Tkačuk, Any pseudo- $pG$ -compact group  $G$  is fine, New Zealand J. Math. 32 (2003) 127–138.
- [6] B.A. Pasynkov, On inverse systems and cardinal functions of topological spaces, Topology Appl. 54 (1993) 97–110.
- [7] B.A. Pasynkov, On relative cellularity of Lindelöf subspaces of topological groups, Topology Appl. 57 (1994) 249–258.
- [8] T. Przymusiński, On the notion of  $n$ -cardinality, Proc. Amer. Math. Soc. 69 (2) (1978) 333–338.
- [9] M.G. Tkačenko, On the Souslin property in free topological groups over compacta, Mat. Zametki 34 (1983) 601–607 (in Russian).
- [10] M.G. Tkačenko, On topologies of free groups, Czechoslovak Math. J. 34 (109) (1984) 541–551.
- [11] M.G. Tkačenko,  $\mathbb{R}$ -factorizable groups and subgroups of Lindelöf  $P$ -groups, Topology Appl. 136 (2004) 135–167.
- [12] V.V. Uspenskii, A topological group generated by Lindelöf  $\Sigma$ -space has the Souslin property, Dokl. Akad. Nauk SSSR 265 (4) (1982) 823–826 (in Russian), also: Soviet Math. Dokl. 26 (1982) 166–169 (in English).
- [13] V.V. Uspenskii, On continuous images of Lindelöf topological groups, Dokl. Akad. Nauk SSSR 285 (4) (1985) 824–827 (in Russian).